Differential Geometry IV: General Relativity

G. Moschidis
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3.1 (a) Let (V_1, m_1) and (V_2, m_2) be two inner product spaces (i.e. $m_i : V_i \times V_i \to \mathbb{R}$ is symmetric and bilinear, but we do not assume that it is non-degenerate). Prove that there exists a unique inner product $m \doteq m_1 \otimes m_2$ on $V_1 \otimes V_2$ with the property that

$$m(X_1 \otimes X_2, Y_1 \otimes Y_2) = m_1(X_1, Y_1) \cdot m_2(X_2, Y_2).$$

(b) Let (V, m) be an inner product space with a non-degenerate inner product m. Prove that m can be extended to a unique non-degenerate inner product on the space of tensors of type (k, ℓ) over V (i.e. the space $\otimes^k V \otimes^\ell V^*$) by the conditions that:

1. $m(f_1 \otimes f_2, g_1 \otimes g_2) = m(f_1, g_1) \cdot m(f_2, g_2)$ for any $f_i, g_i \in \otimes^{k_i} V \otimes^{\ell_i} V^*$, $i = 1, 2$, with $k_1 + k_2 = k$, $\ell_1 + \ell_2 = \ell$,
2. $m(X_\flat, Y_\flat) = m(X, Y)$, where, for any $X \in V$, we define $X_\flat \in V^*$ by $X_\flat \doteq m(X, \cdot)$.

What are the components of this extension of m with respect to a basis of $\otimes^k V \otimes^\ell V^*$ associated to a basis $\{e_a\}_{\alpha=1}^{\dim V}$ of V?

(c) Let (V, m) be as in part (b). Prove that the extension of m to $\otimes^k V \otimes^\ell V^*$ is positive definite if m is positive definite. Is the analogous statement true if m is a Lorentzian inner product?

3.2 Let $\mathcal{M}^n$ be a smooth manifold and let $\omega : \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$ be $C^\infty(\mathcal{M})$-linear functional. We will show that ω is in fact an 1-form on $\mathcal{M}$, i.e. if $Y \in \Gamma(\mathcal{M})$ then, for all $p \in \mathcal{M}$, $\omega(Y)|_p$ depends only on $Y|_p$.

(a) Let $\mathcal{U}$ be an open neighborhood of p covered by a coordinate chart $(x^1, \ldots, x^n)$. Show that there exists an open neighborhood $\mathcal{V}$ of p contained inside $\mathcal{U}$ and smooth vector fields $\{X_i\}_{i=1}^n$ on $\mathcal{M}$ such that $X_i = \frac{\partial}{\partial x^i}$ on $\mathcal{V}$.

(b) Show that if $Y|_p = 0$, then there exists a finite number of vector fields $\{V_k\}_k$ such that

$$Y = \sum_k f_k V_k,$$

where the functions $f_k \in C^\infty(\mathcal{M})$ satisfy $f_k(p) = 0$. Deduce that $\omega(Y)|_p = 0$ and, more generally, $\omega(Y)|_p$ depends only on $Y|_p$.

The same argument should also work for more general $C^\infty(\mathcal{M})$-multilinear maps $T : \Gamma^*(\mathcal{M}) \times \cdots \times \Gamma^*(\mathcal{M}) \times \Gamma(\mathcal{M}) \times \cdots \times \Gamma(\mathcal{M}) \to C^\infty(\mathcal{M})$.

3.3 Let $\mathcal{M}^n$ be a smooth manifold and let $(x^1, \ldots, x^n)$ a local system of coordinates around $p \in \mathcal{M}$. Let also $S \in \otimes^k T_p\mathcal{M} \otimes^l T_p^*\mathcal{M}$ be a tensor of type (k, l) at p and let $S^{i_1 i_2 \ldots i_k}{}_{i_1 j_2 \ldots j_l}$ be its corresponding components. We will define the *contraction* $\mathrm{tr}(S)$ to be the tensor

$$\mathrm{tr}(S) = S^{\alpha i_2 \ldots i_k}{}_{\alpha j_2 \ldots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_l},$$

i.e. the components of $\mathrm{tr}(S)$ in the $(x^1, \dots, x^n)$ coordinates are simply the components of S after summing over the first covariant and contravariant indices. Show that $\mathrm{tr}(S)$ is well-defined *independently* of the choice of coordinate system, i.e. show that if $(y^1, \dots, y^n)$ is a different coordinate system around p and $\tilde{S}^{i_1 i_2 \dots i_k}{}_{j_1 j_2 \dots j_l}$ are the components of S with respect to these coordinates, then

$$\begin{aligned} S^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_l} \\ = \tilde{S}^{\alpha i_2 \dots i_k}{}_{\alpha j_2 \dots j_l} \frac{\partial}{\partial y^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial y^{i_k}} \otimes dy^{i_2} \otimes \dots \otimes dy^{i_l}. \end{aligned}$$

3.4 Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold which is *not* time orientable. Prove that there exists a Lorentzian manifold $(\mathcal{M}', g')$ which is time orientable and a map $F : M' \to M$ which is $2-1$ and a local isometry. Such a space is called a *time-orientable cover.* (*Hint: You might want to consider the causal line seed field $\{X, -X\}$ over $\mathcal{M}$ constructed in Exercise 2.4 last week, and study its properties a a submanifold of $T\mathcal{M}$.*)